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Relating properties of a ring with properties of matrix rings coming from Ornstein dual pairs ☆

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Abstract

In this paper we study the ring-theoretic classification of intermediate rings, $\mathcal{E}_{\alpha\beta}(R)$ of infinite matrix rings, $\mathrm{RCFM}_\alpha(R) \subseteq \mathcal{E}_{\alpha\beta}(R) \subseteq \mathrm{RFM}_\alpha(R)$, from Ornstein's dual pairs (see [D. Ornstein, Dual vector spaces, Ann. Math. 69 (1959) 520–534; A. del Rio, J.J. Simón, Intermediate rings between matrix rings and Ornstein dual pairs, Arch. Math. (Basel) 75 (2000) 256–263]) in case R is semisimple artinian, semiprimary or left or right perfect. To do this, we develop a technique of decomposing an infinite matrix as "infinite sum of submatrices of less size." Then, we show that R is a semisimple artinian ring if and only if $\mathcal{E}_{\alpha\beta}(R)$ is a von Neumann regular ring. We then describe the lattice of finitely generated left ideals, and the lattice of two-sided ideals of $\mathcal{E}_{\alpha\beta}(R)$ by equivalence of idempotents; that is, in classical terms. We also describe the Jacobson radical of $\mathcal{E}_{\alpha\beta}(R)$ for an arbitrary ring R, and then we show, among other results, that R is left perfect if and only if $\mathcal{E}_{\alpha\beta}(R)$ is a semiregular ring.

Introduction and notation

Relating properties of a ring R with properties of infinite matrix rings is an old topic studied by many authors as Jacobson [7], Osofsky [13], O'Meara [11] and others. In this paper we relate some properties of a ring R and infinite matrix rings $\mathcal{E}_{\alpha\beta}(R)$ (see below for definition), for α and β infinite cardinals with $\beta > \omega$, where ω is the first infinite cardinal. The case $\beta = \omega$ has been solved in [2].

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It is a never published known fact that if R is a division ring then $\mathcal{E}_{\alpha\beta}(R)$ is a von Neumann regular ring for some cardinals $\beta > \omega$ (see comment before Corollary 9). We will fill out the ring theoretic classification of $\mathcal{E}_{\alpha\beta}(R)$ (that is, for β being ANY uncountable cardinal), in case R is semisimple artinian, semiprimary or perfect.

In Section 1 we develop techniques to decompose an infinite matrix as "infinite sum of submatrices of less size". As we will see, such decomposition is a kind of partition over a matrix $a \in \mathcal{E}_{\alpha\beta}(R)$ (in fact, it comes from a partition over a subset of α) and it has surprising nice properties. This partition yields new structural information in case $\omega < \beta < \alpha^+$; that is, when $\mathcal{E}_{\alpha\beta}(R)$ is not the row-finite or row- and column-finite matrix ring. We then use this decomposition to see that $\mathcal{E}_{\alpha\beta}(R)$ is a von Neumann regular ring if and only if R is semisimple artinian and $\beta > \omega$, (it is well-known that $\mathcal{E}_{\alpha\omega}(R) = \mathrm{RCFM}_\alpha(R)$, the subring of row and column-finite matrices, is not a von Neumann regular ring for any ring R. See remark prior to Theorem 8). Section 2 is devoted to describe the lattice of finitely generated left ideals, and the lattice of two-sided ideals of $\mathcal{E}_{\alpha\beta}(R)$ by equivalence of idempotents; that is, in classical terms. In Section 3 we extend the results about semisimple rings of Section 1 to more general rings. To do this, we also use our partition to describe the Jacobson radical of $\mathcal{E}_{\alpha\beta}(R)$ for an arbitrary ring R, and then we show, among other results, that R is left perfect if and only if $\mathcal{E}_{\alpha\beta}(R)$ is a semiregular ring.

Throughout this paper, "ring" means associative ring with identity. For any ring R and any infinite cardinal (ordinal-cardinal) α, we denote by $\mathrm{RFM}_\alpha(R)$ the ring of $\alpha \times \alpha$ row-finite matrices over R, and by $\mathrm{B}_\alpha(R)$ (or $\mathrm{RCFM}_\alpha(R)$) the ring of $\alpha \times \alpha$ row and column-finite matrices over R. As in [3, p. 19] or [9, p. 30] we denote by $\omega = \omega_0 = \aleph_0$ the first infinite cardinal, and so $\omega_1 = \aleph_1, \ldots$. Notation and definitions on set theory have been taken from [3] or [9], and following it, for any cardinal α we denote by α^+ the successor cardinal and for any ordinal number α we denote by $\alpha + 1$ the successor ordinal (see [3, p. 19]). For any set A, we denote its cardinality by $|A|$.

Let R be a ring and α an infinite cardinal. For any matrix $a \in \mathrm{RFM}_\alpha(R)$ we denote the (i, j)-entry by $a(i, j)$ and we denote the set of "basic matrices" as e_{ij}; that is, $e_{ij}(h, k) = 1$ if $i = h$ and $j = k$, and 0 otherwise. We set $e_i = e_{ii}$. For any subset $\emptyset \neq X \subseteq \alpha$ we write $e_X = \sum_{x\in X} e_x$; that is $e_X(i, j) = 1$ if $i = j \in X$ and 0 otherwise. In case $X = \emptyset$ we define $e_X = 0$.

The origin of the rings $\mathcal{E}_{\alpha\beta}(R)$ is the notion of some dual pairs of vector spaces studied in [12].

A dual pair (F, N) over a ring R is formed by a left R-module F, a right R-module and a bilinear map $F \times N \to R$ (see [7]). If the bilinear map is nondegenerated on the second variable (as it is always the case in this paper) then N is considered as a submodule of F^*. Conversely a submodule N of F^* defines a dual pair (F, N) over R which is nondegenerated in the second variable. If (F, N) is a dual pair, then F is considered as a topological module with the finite topology induced by N [7]. Let $\mathrm{End}_R^N(F)$ denote the ring of endomorphism of F which are continuous in the N-topology. This ring is also called the ring of endomorphisms with adjoint (see [11]). In general, given a dual pair, a pure matrix description for the endomorphisms with adjoint is not known. There are some cases in which it is possible to give such a description.

Let α and β be two infinite cardinals, so that $\beta \leqslant \alpha^+$. An (α, β)-dual pair over R is a dual pair (F, N) over R such that F is free of rank α and have a basis B so that

$$N = \bigl\{ f \in F^* = \mathrm{Hom}_R(F, R) \mid |B \setminus \mathrm{Ker}\, f| < \beta \bigr\}.$$

We refer to this dual pairs as Ornstein dual pairs. In [15] it is proved that the matrix realization of the ring of endomorphisms with adjoint from an Ornstein dual pair is of the form

$$\mathcal{E}_{\alpha\beta}(R) = \bigl\{ a \in \mathrm{RFM}_\alpha(R) \mid \text{for all } X \subseteq \alpha \text{ such that } |X| < \beta \\ \text{there is } Y \subseteq \alpha \text{ so that } |Y| < \beta \text{ and } e_Y a e_X = a e_X \bigr\}.$$

For example, $\mathcal{E}_{\alpha\omega}(R) = \mathrm{B}_\alpha(R)$ and $\mathcal{E}_{\alpha\alpha^+}(R) = \mathrm{RFM}_\alpha(R)$.

1. Matrix rings over semisimple rings

In this section, we develop techniques to decompose an infinite matrix as "infinite sum of submatrices of less size" as announced in the introduction. Then we will prove that for any infinite cardinal α, the ring $\mathcal{E}_{\alpha\beta}(R)$ is a von Neumann regular ring if and only if R is semisimple artinian and $\beta > \omega$. This means that we have new classes of examples of von Neumann regular rings.

Throughout this section R will be a ring. α and β will be infinite cardinals such that $\omega < \beta$, and γ a subset of α.

Definition 1. Let R, α, β and γ as above. For each $a \in \mathcal{E}_{\alpha\beta}(R)$ we construct:

$$\begin{array}{ll} Y_1^\gamma(a) = \gamma & X_1^\gamma(a) = \bigl\{ i \in \alpha \mid e_i a e_{Y_1^\gamma(a)} \neq 0 \bigr\} \\ \vdots & \vdots \\ Y_n^\gamma(a) = \bigl\{ j \in \alpha \mid e_{X_{n-1}^\gamma(a)} a e_j \neq 0 \bigr\} & X_n^\gamma(a) = \bigl\{ i \in \alpha \mid e_i a e_{Y_n^\gamma(a)} \neq 0 \bigr\} \\ \vdots & \vdots \end{array}$$

and define,

$$c_\gamma(a) = \bigcup_{n \in \mathbb{N}} Y_n^\gamma(a), \qquad r_\gamma(a) = \bigcup_{n \in \mathbb{N}} X_n^\gamma(a).$$

If $\gamma = \{i\}$ we only write $r_i(a)$ and $c_i(a)$.

Remark 2. Note that $X_1^\gamma(a) \subseteq \cdots \subseteq X_n^\gamma(a) \subseteq \cdots$ and $Y_2^\gamma(a) \subseteq \cdots \subseteq Y_n^\gamma(a) \subseteq \cdots$; but $Y_1^\gamma(a)$ is not necessarily a subset of $Y_2^\gamma(a)$. As an example consider $ae_\gamma = 0$.

We will study some properties of this construction. We begin by showing some arithmetical properties.

Proposition 3. *Let R, α, β and γ as above. If $a \in \mathcal{E}_{\alpha\beta}(R)$ then*

(1) *For all $x \in \alpha$, $e_x a e_{c_\gamma(a)} \neq 0$ if and only if $x \in r_\gamma(a)$.*
(2) *For all $x \in \alpha$, $e_{r_\gamma(a)} a e_x \neq 0$ if and only if $x \in c_\gamma(a)$. Thus,*
(3) $e_{r_\gamma(a)} a = e_{r_\gamma(a)} a e_{c_\gamma(a)} = a e_{c_\gamma(a)}$

Proof. Set $r_\gamma = r_\gamma(a)$, $c_\gamma = c_\gamma(a)$, $X_n^\gamma = X_n^\gamma(a)$ and $Y_n^\gamma = Y_n^\gamma(a)$.

(1) If $e_x a e_{c_\gamma} \neq 0$ then there is $n \in \mathbb{N}$ such that $e_x a e_{Y_n^\gamma} \neq 0$, so that $x \in X_n^\gamma$.
(2) Analogous to (1).
(3) Consider any $x \in \alpha$. If $x \notin r_\gamma$ then $e_x e_{r_\gamma} a = 0$ and $e_x a e_{c_\gamma} = 0$ (by (1)). Hence, $e_x e_{r_\gamma} a = 0 = e_x a e_{c_\gamma}$. If $x \in r_\gamma$ then $x \in X_n^\gamma$, for some $n \in \mathbb{N}$. Thus $e_x a = e_x a e_{Y_{n+1}^\gamma}$ and this implies that $e_x e_{r_\gamma} a = e_x a e_{c_\gamma}$. Hence $e_{r_\gamma} a = a e_{c_\gamma}$. □

Next proposition is related with operations in set theory.

Proposition 4. *Let R, α, β and γ as above. If $a \in \mathcal{E}_{\alpha\beta}(R)$ then*

(1) *If $\gamma' \subseteq \gamma$ then $c_{\gamma'}(a) \subseteq c_\gamma(a)$ and $r_{\gamma'}(a) \subseteq r_\gamma(a)$.*
(2) *If $t \in c_\gamma(a)$ then there exists $k \in \gamma$ such that $c_t(a) = c_k(a)$.*
(3) *If $t \in r_\gamma(a)$ then there exists $k \in \gamma$ such that $t \in r_k(a) \subseteq r_\gamma(a)$.*
(4) *Let $i, j \in \alpha$ such that $i \neq j$. Then the following conditions are equivalent*:
   (a) $c_i(a) \cap c_j(a) \neq \emptyset$.
   (b) $c_i(a) = c_j(a)$.
   (c) $r_i(a) = r_j(a) \neq \emptyset$.
   (d) $r_i(a) \cap r_j(a) \neq \emptyset$.
(5) *The relation $i \sim j$ if and only if $c_i(a) \cap c_j(a) \neq \emptyset$ is an equivalence relation on α.*
(6) $c_\gamma(a) = \bigcup_\gamma c_i(a) = \bigcup_{c_\gamma(a)} c_j(a)$ *and* $r_\gamma(a) = \bigcup_\gamma r_i(a) = \bigcup_{r_\gamma(a)} r_j(a)$.

Proof. Set $r_\gamma = r_\gamma(a)$, $c_\gamma = c_\gamma(a)$, $X_n^\gamma = X_n^\gamma(a)$ and $Y_n^\gamma = Y_n^\gamma(a)$.

(1) It is trivial.
(2) Suppose that $t \in c_\gamma$. Then there is a $n \in \mathbb{N}$ such that $t \in Y_n^\gamma$. If $n = 1$ we are done, by (1). If $n > 1$, there is an $i_1 \in X_{n-1}^\gamma$ such that $e_{i_1} a e_t \neq 0$. This, in turn implies that there is a $j_1 \in Y_{n-1}^\gamma$ such that $e_{i_1} a e_{j_1} \neq 0$. We proceed in this way up to $i_{n-1} \in X_1^\gamma$ and $j_{n-1} \in Y_1^\gamma = \gamma$ such that $e_{i_{n-1}} a e_{j_{n-1}} \neq 0$. Set $k = j_{n-1}$. Then $t \in Y_n^k$, and so $Y_m^t \subseteq Y_{n+m}^k$, which implies $c_t(a) \subseteq c_k(a)$. A recursive computation shows that $k \in Y_n^t$, and hence $c_k(a) \subseteq c_t(a)$.
(3) Take any $t \in r_\gamma$. Then $t \in X_n^\gamma$. This means $e_{Y_n^\gamma} a e_t \neq 0$. We proceed as in the proof of (2) to pick up $j_n \in Y_1^\gamma$ and $i_{n-1} \in X_1^\gamma$; with $e_{i_{n-1}} a e_{j_n} \neq 0$. Setting $k = j_n$, we have that $t \in r_k(a) \subseteq r_\gamma$.
(4) [a $\Rightarrow$ b] Immediate from (2).
   [b $\Rightarrow$ c] Since $i \neq j$ then r_i and r_j are not empty, by construction. The equality follows from Proposition 3.

[c ⇒ d] Is obvious.
[d ⇒ a] Suppose $r_i(a) \cap r_j(a) \neq \emptyset$ and take any $t \in r_i(a) \cap r_j(a)$. Then $t \in X_n^i$ and $t \in X_m^j$, for some $n, m \in \mathbb{N}$. Set $T = \{j \in \alpha \mid e_t a e_j \neq 0\}$. Then $\emptyset \neq T \subseteq Y_n^i \cap Y_m^j$, so that $c_i(a) \cap c_j(a) \neq \emptyset$.

The remaining proofs are obvious. □

Finally, the following properties are related with cardinality.

Proposition 5. *Let R, α, β and γ as above. If $a \in \mathcal{E}_{\alpha\beta}(R)$ then*

(1) *If $Y_{n+1}^\gamma(a)$ or $X_n^\gamma(a)$ is infinite then $|Y_{n+t+1}^\gamma(a)| \leqslant |X_{n+t}^\gamma(a)|$ for all $t \in \mathbb{N}$. In particular, if $|r_\gamma(a)| > \omega$, then $|c_\gamma(a)| \leqslant |r_\gamma(a)|$.*
(2) *If $X_{n+1}^\gamma(a)$ is infinite then $|X_n^\gamma(a)| < |X_{n+1}^\gamma(a)|$ if and only if, there is $k \in Y_{n+1}^\gamma(a)$ such that $|X_n^\gamma(a)| < |\{i \in \alpha \mid e_i a e_k \neq 0\}|$.*
(3) *Suppose $\omega < \beta$. If $|\gamma| < \beta$ then $|c_\gamma(a)|, |r_\gamma(a)| < \beta$.* (*Even if β is an accessible cardinal*!)

Proof. Set $r_\gamma = r_\gamma(a)$, $c_\gamma = c_\gamma(a)$, $X_n^\gamma = X_n^\gamma(a)$ and $Y_n^\gamma = Y_n^\gamma(a)$.

(1) It is a direct consequence of the fact that Y_{n+1}^γ is a union of finite sets, and $|X_n^\gamma| \leqslant |X_{n+1}^\gamma|$.
(2) Suppose that $|X_n^\gamma| < |X_{n+1}^\gamma|$. If X_n^γ is a finite set, we are done. Otherwise, by (1) above, we know that $|Y_{n+1}^\gamma| \leqslant |X_n^\gamma|$. If $|\{i \in \alpha \mid e_i a e_j \neq 0\}| \leqslant |X_n^\gamma|$ for all $j \in Y_{n+1}^\gamma$ then $|X_{n+1}^\gamma| \leqslant |X_n^\gamma|$, which is impossible, by hypothesis.
(3) Both $|X_n^\gamma|$ and $|Y_n^\gamma|$ are less than β by definition of $\mathcal{E}_{\alpha\beta}(R)$. Clearly, $|c_\gamma| = \beta$ implies $|r_\gamma| = \beta$.
Suppose that $|r_\gamma| = \beta$. Then we may find a (countable) increasing sequence of cardinals $\omega \leqslant |X_{n_1}^\gamma| < \cdots < |X_{n_m}^\gamma| < \cdots$ whose limit is β. By (2) above, we may find a countable set of columns, say $C = \{k_1, \ldots, k_m, \ldots\}$, such that $|\{i \in \alpha \mid e_i a e_C \neq 0\}| = \beta$, contradicting the definition of $\mathcal{E}_{\alpha\beta}$ (recall that $\omega < \beta$). □

Remark 6. We note that the hypothesis $\omega < \beta$ in Proposition 5.3 is not superfluous. Let a be the matrix

$$a(i, j) = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i = j + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $a \in \mathcal{E}_{\alpha\beta}(R)$ with $\beta = \omega$. Let $\gamma \in \alpha$. If γ is a limit cardinal then $r_\gamma(a) = c_\gamma(a) = \{\gamma\}$. If γ in not a limit, the set $S_\gamma = \{\delta \in \alpha \mid \delta + n = \gamma \text{ for some } n < \omega\}$ is not empty and then there is a first element δ, which is a limit ordinal. In this case, $r_\gamma(a) = c_\gamma(a) = \{\delta + 1, \ \delta + 2, \ \ldots\}$.

We are now ready to show a decomposition.

Lemma 7. *Let R be a ring. Let α, β be infinite cardinals and let $a \in \mathcal{E}_{\alpha\beta}(R)$. Consider the set $\Upsilon_a = \{i \in \alpha \mid r_i(a) \neq \emptyset\}$ and consider the equivalence relation on Υ_a, given by $i \sim j \Leftrightarrow c_i(a) = c_j(a) \Leftrightarrow r_i(a) = r_j(a)$ (see Proposition 4). Let Δ_a be a complete set of representatives. For each pair $(r_j(a), c_j(a))$ where $j \in \Delta_a$, set $t_j(a) = r_j(a) \cup c_j(a)$. Then*:

(1) *The matrix $e_{t_j(a)} a e_{t_j(a)}$ may be viewed as an element of* $\mathrm{RFM}_{t_j(a)}(R)$.
(2) *If $\omega < \beta < \alpha^+$ then $|t_j(a)| < \beta$.*
(3) $a = \sum_{j \in \Delta_a} e_{r_j(a)} a e_{c_j(a)}$.

Proof. (1) It is obvious. (2) and (3) follow from Proposition 4. □

We mentioned in the introduction that, for any ring R and any infinite cardinal α, $\mathcal{E}_{\alpha\omega}(R)$ is not a von Neumann regular ring. To see this, take the matrix $a \in \mathcal{E}_{\alpha\omega}(R)$ defined in Remark 6. A direct computation shows that the matrix x, defined as

$$x(i, j) = \begin{cases} 1 & \text{if } i = j, \\ 1 & \text{if } i \text{ is not a limit ordinal and } i = j + n \text{ with } n < \omega, \\ 0 & \text{otherwise,} \end{cases}$$

verifies $x = a^{-1}$ and $x \in \mathcal{E}_{\alpha\omega_1}(R) \setminus \mathcal{E}_{\alpha\omega}(R)$. So the equation $aya = a$ have not solution in $\mathcal{E}_{\alpha\omega}(R)$.

Next theorem is our main result. It shows, in particular, that the situation described above does not happen in case $\omega < \beta$ and proves that for any $a \in \mathcal{E}_{\alpha\omega}(R)$, the equation $aya = a$ has certainly a solution in $\mathcal{E}_{\alpha\omega_1}(R)$.

Theorem 8. *Let R be a ring with identity and α and β be infinite cardinals. R is a semisimple artinian ring and $\beta > \omega$ if and only if $\mathcal{E}_{\alpha\beta}(R)$ is a von Neumann regular ring.*

Proof. Suppose first that R is a semisimple artinian ring and $\beta > \omega$. Take any $a \in \mathcal{E}_{\alpha\beta}(R)$. We set

$$r = \{i \in \alpha \mid e_i a = 0\} \quad \text{and} \quad c = \{j \in \alpha \mid a e_j = 0\}.$$

Let Υ_a, Δ_a as in Lemma 7 and set $\Upsilon = \Upsilon_a$ and $\Delta = \Delta_a$.

For each $j \in \Delta$, let $r_j(a), c_j(a), t_j(a)$ be as in Lemma 7 and set $r_j = r_j(a)$, $c_j = c_j(a)$ and $t_j = t_j(a)$. Recall that $|t_j| < \beta$. Consider the matrix $e_{r_j} a e_{c_j}$. We know that it may be viewed as an element of the ring of matrices $\mathrm{RFM}_{t_j}(R)$, which we know that it is a von Neumann regular ring (see [16]). Thus, there exists an element $x_j \in \mathrm{RFM}_{t_j}(R)$, such that $e_{r_j} a e_{c_j} = e_{r_j} a e_{c_j} x_j e_{r_j} a e_{c_j}$. Now, let $y_j = e_{c_j} x_j e_{r_j}$, viewed as an element of $\mathrm{RFM}_\alpha(R)$. It is clear that $e_{r_j} a e_{c_j} = e_{r_j} a e_{c_j} y_j e_{r_j} a e_{c_j}$. As we have chosen r_j, being not empty and distinct, then by Proposition 4 the sum $y = \sum_{j \in \Delta} y_j$ makes (obvious) sense, and, moreover, a direct computation shows:

(a) $y = \sum_{i\in\Delta} e_{c_i} y \sum_{i\in\Delta} e_{r_i}$,
(b) $e_{c_i} y = y e_{r_j} = e_{c_i} y e_{r_j}$.

We claim that $y \in \mathcal{E}_{\alpha\beta}(R)$. Let $X \subseteq \alpha$, be such that $|X| < \beta$. If $x \in X$ verifies $x \notin \bigcup_{i\in\Delta} r_i$ then $ye_x = 0$, by construction, then we may suppose WLOG that $X \subseteq \bigcup_{j\in\Delta} r_j$. Set $\Lambda = \{j \in \alpha \mid r_j \cap X \neq \emptyset\}$. By Proposition 4, we see that each element of X belongs to only one of the r_j's; so that $|\Lambda| \leqslant |X| < \beta$; hence by Proposition 5, $|r_\Lambda|, |c_\Lambda| < \beta$.

Now $ye_x = ye_{r_\Lambda} e_x = e_{c_\Lambda} y e_x$ by Property (b) above, and hence $y \in \mathcal{E}_{\alpha\beta}$.

Finally, we shall see that $a = aya$. Take $i \in \alpha$. If $i \in r$ then $e_i a = 0 = e_i aya$. If $e_i a \neq 0$, then $i \in r_j$ for some suitable representative $j \in \alpha$. By Proposition 3 and the fact that $e_{r_j} a e_{c_j} = e_{r_j} a e_{c_j} y_j e_{r_j} a e_{c_j}$, we have,

$$\begin{aligned} e_i aya &= e_i e_{r_i} aya = e_i e_{r_i} a e_{c_j} ya = e_i e_{r_i} a e_{c_j} y_j a = e_i e_{r_i} a e_{c_j} y_j e_{r_j} a \\ &= e_i e_{r_i} a e_{c_j} y_j e_{r_j} a e_{c_j} = e_i e_{r_j} a e_{c_j} = e_i e_{r_j} a = e_i a. \end{aligned}$$

Conversely, suppose WLOG, $\beta \leqslant \alpha^+$. By the remark prior to the theorem, we know that $\mathrm{B}_\alpha(R) = \mathcal{E}_{\alpha\omega}(R)$ is not von Neumann regular for all ring R, so that $\beta > \omega$. We may identify $e_\beta \mathcal{E}_{\alpha\beta}(R) e_\beta = \mathcal{E}$ as $\mathrm{RFM}_\beta(R)$. Take any $x \in \mathcal{E}$. Then $x = e_\beta x e_\beta$, and $x = xax$, for some $a \in \mathcal{E}_{\alpha\beta}(R)$. Setting $a' = e_\beta a e_\beta$ we have $x = xa'x$ and $a' \in \mathcal{E}$, so it may be viewed as an element of $\mathrm{RFM}_\beta(R)$. This means that $\mathrm{RFM}_\beta(R)$ is a von Neumann regular ring and so by [16] R is a semisimple artinian ring. □

There are relations between properties of a ring R and infinite matrix rings over R that do not depend on cardinality. An example is primitivity. If a ring R has a faithful simple left module M then $\mathrm{Hom}_R(R^{(\alpha)}, M)$ has structure of left $\mathcal{E}_{\alpha\beta}(R)$-module for any β and, as such, it is a simple module.

In [12, Definition 2], D. Ornstein divides the cardinals into tree classes A, B and C. The following corollary is related with Problem 54 in [4]. It was solved in 1995 by Professor Luca Giudici, in a private communication to Professor K.R. Goodearl. A sketch of the proof is as follows.

Let (F, N) be a dual pair of vector spaces. A subspace (of F or N) is closed if and only if it is equal to its double annihilator (in the usual sense). The dual pair (F, N) is called modular if in either F or N the sum of any two closed subspaces is closed. The dual pair (F, N) is called splittable if in either F or N every closed subspace admits a closed complement such that the annihilators span the other subspace.

The subspaces of F which are closed form a complete irreducible DAC lattice, L, in the sense of [10]. Theorem 4.1 in [12] shows that, when (F, N) is an (α, β)-dual pair over R and β belongs to class A, the lattice L is complemented and modular. However, L is not upper or lower continuous. So the regular ring that coordinatizes L, gives the counterexample.

Then it is made the observation that the von Neumann regular ring that coordinatizes the lattice of closed subspaces of F can be represented as the ring of continuous endomorphisms of F. See [14]. That this ring is a Baer ring follows from a remark by Kaplansky (see also [6]). Finally, the fact that the ring is von Neumann regular comes from [5, Theorem 2, p. 184].

In our context, the proof of this result comes immediately from Theorem 8, the remark by Kaplansky and Corollary 13.20 of [4] together with the fact that $\mathcal{E}_{\alpha\beta}(D)$ is indecomposable, for any division ring D.

Corollary 9. *If R is a semisimple ring and α, β are infinite cardinals such that β belongs to class A, then $\mathcal{E}_{\alpha\beta}(D)$ is a von Neumann regular ring, and a Baer ring which is neither right nor left continuous.*

2. Equivalence of idempotents

Idempotents and finitely generated projective modules plays an important role in the study of von Neumann regular rings. In this section, we develop some tools to study the lattice of finitely generated left (right) ideals, and the lattice of two-sided ideals of $\mathcal{E}_{\alpha\beta}(R)$ when R is a semisimple artinian ring.

Recall from [8] that for any ring with identity R two idempotents $e, f \in R$ are equivalent if and only if $Re \cong Rf$. See also [1].

Next theorem is known in the cases of row-finite matrices, see [20], and row- and column-finite matrices, see [12,18].

Theorem 10. *Let R be a semisimple ring, α and β infinite cardinals. Two idempotents $f, g \in \mathcal{E}_{\alpha\beta}(R)$ are equivalent if and only if $R^{(\alpha)} f \cong R^{(\alpha)} g$.*

Proof. By our comment above we only consider the case $\omega < \beta$. By results on Morita equivalences in [15] we only have to prove this for $R = D$, a division ring. It is well-known that for any infinite set γ and for any idempotent $f \in \mathrm{RFM}_\gamma(D)$, there is a subset $\delta \subset \gamma$ and a diagonal matrix d, where $d(i,i) = 1$ if and only if $i \in \delta$, such that f and d are equivalent idempotents (in $\mathrm{RFM}_\gamma(D)$).

First we will prove that for any idempotent $f \in \mathcal{E}_{\alpha\beta}(D)$, there exists a diagonal matrix d whose entries are all 0 or 1; such that f and d are equivalent idempotents in $\mathcal{E}_{\alpha\beta}(D)$.

Let $\Upsilon = \Upsilon_f$ and $\Delta = \Delta_f$ as in Lemma 7.

Suppose first that Δ is an infinite set. Let P be a partition of Δ such that, for each $\gamma \in P$ we have $|\gamma| = \omega$. For each pair $(r_\gamma(f), c_\gamma(f))$ where $\gamma \in P$, we set $r_\gamma = r_\gamma(f)$, $c_\gamma = c_\gamma(f)$ and $t_\gamma = r_\gamma \cup c_\gamma$. By Proposition 5, we get $|t_\gamma| = |r_\gamma| < \beta$.

For each $\gamma \in P$, we consider the matrix $e_{r_\gamma} f e_{c_\gamma}$. As usual, we view this matrix as an element of the ring of matrices $\mathrm{RFM}_{t_\gamma}(D)$. By Proposition 4, we have that $e_{r_\gamma} f e_{c_\gamma} e_{r_\gamma} f e_{c_\gamma} = e_{r_\gamma} f e_{c_\gamma}$; so that, $e_{r_\gamma} f e_{c_\gamma}$ may be viewed as an idempotent in $\mathrm{RFM}_{t_\gamma}(D)$.

Since $|\gamma| = \omega$ then $|c_\gamma| \leqslant |r_\gamma| = |t_\gamma|$. By basic linear algebra we know that there exists a diagonal matrix $d_\gamma \in \mathrm{RFM}_{t_\gamma}(D)$ whose entries belong to $\{0, 1\}$, and such that $d_\gamma = d_\gamma e_{r_\gamma} = e_{r_\gamma} d_\gamma$, and there also exist matrices u_γ, v_γ such that

$$u_\gamma = d_\gamma u_\gamma e_{r_\gamma} f e_{c_\gamma} \quad \text{and} \quad v_\gamma = e_{r_\gamma} f e_{c_\gamma} v_\gamma d_\gamma,$$
$$u_\gamma v_\gamma = d_\gamma \quad \text{and} \quad v_\gamma u_\gamma = e_{r_\gamma} f e_{c_\gamma} v_\gamma.$$

Now we may consider each of u_γ, v_γ, d_γ as an element of $\mathcal{E}_{\alpha\beta}(D)$. For every two different elements γ and γ' in P, one has $d_\gamma d_{\gamma'} = 0$; $u_\gamma v_{\gamma'} = u_\gamma e_{r_\gamma} f e_{c_\gamma} e_{r_{\gamma'}} f e_{c_{\gamma'}} v_{\gamma'} = u_\gamma e_{r_\gamma} f e_{c_{\gamma'}} v_{\gamma'} = u_\gamma f e_{c_\gamma} e_{c_{\gamma'}} v_{\gamma'} = 0$ by Proposition 3, and $v_{\gamma'} u_\gamma = v_{\gamma'} d_{\gamma'} d_\gamma u_\gamma = 0$.

Then, the sums $d = \sum_{\gamma\in P} d_\gamma$, $u = \sum_{\gamma\in P} u_\gamma$, and $v = \sum_{\gamma\in P} v_\gamma$, make sense and it is easy to see that $u, v, d \in \mathcal{E}_{\alpha\beta}(D)$, $uv = d$, and $vu = f$.

Suppose that Δ is a finite set. Then $f = e_{r_\Delta} f e_{c_\Delta}$, so that f may be viewed as an element of $\mathrm{RFM}_{t_\Delta}(D)$ and so we may proceed as in previous paragraph.

In any case, $u, v, d \in \mathcal{E}_{\alpha\beta}(D)$, $uv = d$, and $vu = f$.

It is well-known (see, for example, [7]) that if $\mathcal{E}_{\alpha\beta}(D)f \cong \mathcal{E}_{\alpha\beta}(D)d$ as left ideals, then $D^{(\alpha)} f \cong D^{(\alpha)} d$, viewing f and d as endomorphisms of $D^{(\alpha)}$. From here, the result is obvious. □

As an immediate consequence of the theorem above we have the following corollary.

Corollary 11. *Let R be a semisimple ring, and α and β infinite cardinals such that $\omega < \beta$. The lattice of two-sided ideals of $\mathcal{E}_{\alpha\beta}(R)$ is linearly ordered by inclusion and every two-sided ideal is a left projective ideal.*

3. Jacobson radical and semiregular matrix subrings

We denote the Jacobson radical of a ring R by $J(R)$. The Jacobson radical of a row-finite matrix ring has been described in [17] (see also [19]). In the case of row and column-finite matrices, it is proved in [18] that $J(\mathrm{RCFM}(R)) = J(\mathrm{RFM}(R)) \cap J(\mathrm{CFM}(R))$, where $\mathrm{CFM}(R)$ denotes the column-finite matrix ring.

Lemma 12. *Let R be a ring, and α and β cardinals such that $\omega < \beta$. Let $a \in J(\mathrm{RFM}_\alpha(R)) \cap \mathcal{E}_{\alpha\beta}(R)$ and consider the element $1 - a$. In the setting of Definition 1, for any $\gamma \subseteq \alpha$, $r_\gamma(1 - a) = c_\gamma(1 - a)$.*

Proof. By [17], we know that $J(\mathrm{RFM}_\alpha(R)) \subseteq \mathrm{RFM}_\alpha(J(R))$. Since $a \in J(\mathrm{RFM}_\alpha(R)) \cap \mathcal{E}_{\alpha\beta}(R)$ then there are no units in the diagonal of a. Then for any $i \in \alpha$ we have $e_i(1 - a)e_i \neq 0$. From this, using notation from Definition 1, it is easy to see that if $k \in r_\gamma(1 - a)$ then there exists $n \in \mathbb{N}$ such that $k \in X_n^\gamma(1 - a)$ and then $k \in Y_{n+1}^\gamma(1 - a)$, because $e_k(1 - a)e_k \neq 0$. Conversely, if $k \in c_\gamma(1 - a)$ then there is some $n \in \mathbb{N}$ such that $k \in Y_n^\gamma(1 - a)$, so that $k \in X_n^\gamma(1 - a)$. □

Next result is a little surprising.

Theorem 13. *Let R be a ring and α, β cardinals such that $\omega < \beta$. Then $J(\mathcal{E}_{\alpha\beta}(R)) = J(\mathrm{RFM}_\alpha(R)) \cap \mathcal{E}_{\alpha\beta}(R)$.*

Proof. First, we prove that $J(\mathcal{E}_{\alpha\beta}(R)) \subset J(\mathrm{RFM}_\alpha(R))$. Take any $a \in J(\mathcal{E}_{\alpha\beta}(R))$. We shall prove that the family of left ideals generated by the columns of a is right vanishing. By

[17] or by [19], this implies that $a \in J(\mathrm{RFM}_\alpha(R))$. Let us denote the jth column of a by $a(j)$.

Take distinct columns $a(j_1), \ldots, a(j_n), \ldots$ and a sequence $\{s_i\}_{i\in\mathbb{N}}$ of elements of R, such that $s_i \in R^{(\alpha)}a(j_i)$ (row by column product). There exist elements in $R^{(\alpha)}$, $u_1, \ldots, u_n, \ldots$ such that $s_i = u_i a(j_i)$. Let u be the matrix whose first ω rows are those u_i. Note that $u \in \mathcal{E}_{\alpha\beta}(R)$, because $\omega < \beta$. Since ua is a row-finite matrix, there exists a sequence $n_1 < n_2 < \ldots$ of natural numbers such that $u_{n_j} a(n_{j+1}) = 0$.

Let $r_1, r_2, \ldots$ be any sequence in R. Let $v \in \mathcal{E}_{\alpha\beta}(R)$ be the matrix whose first ω rows are those u_{n_j} and let $w \in \mathcal{E}_{\alpha\beta}(R)$ be the matrix defined by $w(n_i, i+1) = r_i$, and 0 otherwise.

It is clear that $vaw \in J(\mathcal{E}_{\alpha\beta}(R))$. Note that vaw has the form

$$\begin{pmatrix} 0 & s_1 r_1 & 0 & \ldots & \\ 0 & x_{22} & s_{n_2} r_2 & 0 & \ldots \\ \vdots & & \ddots & & \end{pmatrix}.$$

From this point, the argument of this proof is analogous to that of [19], however we will write it for the convenience of the reader.

Consider the matrix

$$\begin{pmatrix} 1 & -s_1 r_1 & 0 & \ldots & \\ 0 & 1 & -s_{n_2} r_2 & 0 & \ldots \\ \vdots & & \ddots & & \end{pmatrix}.$$

It is clear that $vaw + b$ is an invertible matrix in $\mathrm{RFM}_\alpha(R)$. Let $u = (vaw + b)^{-1}$; then $bu = 1 - vawu$, and note that $vawu \in J(\mathcal{E}_{\alpha\beta}(R))$. Thus bu is invertible in $\mathrm{RFM}_\alpha(R)$. By [1, Exercise 28.1] this implies that there is a $k \in \mathbb{N}$ such that $s_{n_1} r_1 \cdots s_{n_k} r_k = 0$. Since this happens for any sequence $\{r_j\}$, it may be chosen so that $s_1 \cdots s_{n_k} = 0$.

Conversely, we have to prove that for any $a \in J(\mathrm{RFM}_\alpha(R)) \cap \mathcal{E}_{\alpha\beta}(R)$, the matrix $(1-a)^{-1}$ belongs to $\mathcal{E}_{\alpha\beta}(R)$. To see this consider $\Upsilon = \Upsilon_{(1-a)}$, $\Delta = \Delta_{(1-a)}$, and $r_j = r_j(1-a), c_j = c_j(1-a), t_j = t_j(1-a)$ as in Lemma 7. By Lemma 12, we know that $r_i = c_i$. Moreover, $e_{r_i} a e_{c_i}$ may be viewed as element of $\mathrm{RFM}_{r_i}(R)$ and, as such, it belongs to $J(\mathrm{RFM}_{r_i}(R))$. So that $(e_{r_i} - e_{r_i} a e_{c_i})$ is invertible in $\mathrm{RFM}_{r_i}(R)$. Set $(e_{r_i} - e_{r_i} a e_{c_i})^{-1} = b_i$ (recall that $r_i = c_i$). It is clear now that the sum $\sum b_i$ over all representatives makes sense, belongs to $\mathcal{E}_{\alpha\beta}(R)$ and $\sum b_i = (1-a)^{-1}$. □

By [18, Corollary 4.3] a ring R which is left perfect but not right perfect shows that our theorem above is false in case $\beta = \omega$.

Using the description of the radical given above, we may repeat arguments in [17,19] to obtain the following results on T-nilpotence.

Corollary 14. *Let R be a ring, and α and β cardinals such that $\omega < \beta$. Then $J(\mathcal{E}_{\alpha\beta}(R)) = \mathcal{E}_{\alpha\beta}(J(R))$ if and only if $J(R)$ is left T-nilpotent.*

Corollary 15. *Let R be a ring, and α and β cardinals such that $\omega < \beta$. Then the following conditions are equivalent*:

(1) *$J(\mathcal{E}_{\alpha\beta}(R))$ is left T-nilpotent.*
(2) *$J(R)$ is nilpotent.*
(3) *$J(\mathcal{E}_{\alpha\beta}(R))$ is nilpotent.*

Proposition 16. *Let R be a ring, and α and β infinite cardinals such that $\omega < \beta$. If $\mathcal{E}_{\alpha\beta}(R)/J(\mathcal{E}_{\alpha\beta}(R))$ is a von Neumann regular ring then $J(R)$ is left T-nilpotent.*

Proof. Let $a_1, \dots, a_n, \dots$ be a sequence in $J(R)$, and consider the diagonal matrix $a(k,k) = a_k$ for all $k < \omega$ and 0 otherwise. By hypothesis, there is an element $b \in \mathcal{E}_{\alpha\beta}(R)$ such that $a - aba \in J(\mathcal{E}_{\alpha\beta}(R))$. Now an easy computation shows that, for any $k < \omega$, $(a - aba)(k,k) = a_k b(k,k) a_k$ and $(a - aba)(j,j) = 0$ if $j \geqslant \omega$. Since $a_k \in J(R)$ for each $k \in \mathbb{N}$, there is an $u_k \in J(R)$ such that $u_k(1 - a_k b(k,k)) = 1$. Thus $u_k(a_k - a_k b(k,k) a_k) = a_k$. Let u be the diagonal matrix $u(k,k) = u_k$ if $k < \omega$ and 0 otherwise. Again an easy computation shows that $u(a - aba)(r,r) = a_k$. Since $u(a - aba) \in J(\mathcal{E}_{\alpha\beta}(R))$, by Theorem 13 there is a $n \in \mathbb{N}$ such that $a_1 \cdots a_n = 0$. □

Theorem 17. *Let R be a ring, and α and β infinite cardinals. The following conditions are equivalent*:

(1) *R is a left perfect ring and $\omega < \beta$.*
(2) *$\mathcal{E}_{\alpha\beta}(R)/J(\mathcal{E}_{\alpha\beta}(R))$ is a von Neumann regular ring.*
(3) *$\mathcal{E}_{\alpha\beta}(R)$ is a semiregular ring.*
(4) *$J(\mathcal{E}_{\alpha\beta}(R)) = \mathcal{E}_{\alpha\beta}(J(R))$ and $\mathcal{E}_{\alpha\beta}(R/J(R))$ is von Neumann regular ring.*

Proof. [$1 \Rightarrow 2$] It follows immediately from the fact that $\mathcal{E}_{\alpha\beta}(R)/J(\mathcal{E}_{\alpha\beta}(R)) \cong \mathcal{E}_{\alpha\beta}(R/J(R))$, together with Theorem 8.

[$3 \Rightarrow 1$] Suppose $\mathcal{E}_{\alpha\beta}(R)$ is a semiregular ring for some α, β infinite cardinals. By Proposition 16 and Corollary 14, $\mathcal{E}_{\alpha\beta}(R/J(R))$ is a von Neumann regular ring, and by Theorem 8 we must have that $\omega < \beta$ and that $R/J(R)$ is a semisimple ring. So R is left perfect.

[$1 \Rightarrow 4$] By implications above we only have to see that idempotents lift modulo $J(\mathcal{E}_{\alpha\beta}(R))$. By hypothesis $J(\mathcal{E}_{\alpha\beta}(R)) = \mathcal{E}_{\alpha\beta}(J(R))$, so we may identify $\mathcal{E}_{\alpha\beta}(R)/J(\mathcal{E}_{\alpha\beta}(R))$ and $\mathcal{E}_{\alpha\beta}(R/J(R))$. Take any idempotent $f \in \mathcal{E}_{\alpha\beta}(R)/J(\mathcal{E}_{\alpha\beta}(R))$. It is well-known that $(R/J(R))^{(\alpha)} \cdot f$ has a projective cover in R-mod, and by properties of decomposition of left projective modules over left perfect rings, we may find an idempotent $g \in \mathrm{RCFM}_\alpha(R) \subset \mathcal{E}_{\alpha\beta}(R)$ such that $R^{(\alpha)}g$ is a left projective cover of $(R/J(R))^{(\alpha)} \cdot f$; that is $(R/J(R))^{(\alpha)} \cdot f \cong (R/J(R))^{(\alpha)} \cdot g$. By Theorem 10, $\mathcal{E}_{\alpha\beta}(R/J(R))f \cong \mathcal{E}_{\alpha\beta}(R/J(R))g$; hence f lifts to g modulo $J(\mathcal{E}_{\alpha\beta}(R))$.

The proofs of all the remaining implications are trivial. □

Our last result is an immediate consequence of Corollary 15.

Corollary 18. *Let R be a ring and α and β infinite cardinals. The following conditions are equivalent*:

(1) *R is a semiprimary ring and $\omega < \beta$.*
(2) *$J(\mathcal{E}_{\alpha\beta}(R))$ is T-nilpotent and $\mathcal{E}_{\alpha\beta}(R/J(R))$ is a von Neumann regular ring.*
(3) *$J(\mathcal{E}_{\alpha\beta}(R))$ is nilpotent and $\mathcal{E}_{\alpha\beta}(R/J(R))$ is a von Neumann regular ring.*

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